THE MOTIONS OF THE ATMOSPHERE AND ESPECIALLY ITS WAVES.*

Address delivered before the Meteorological Section of the Association of German Naturalists and Physicians at the Annual Meeting held in Vienna, September 25, 1894.

BY DR. E. HERRMANN.

The inadmissibility of those views according to which the motions of the atmosphere consist in the development of independent cyclones and anti-cyclones is, of late years, more and more plainly recognized. This conclusion has been arrived at, not so much through a severe criticism of the fundamental basis upon which these erroneous views had been established, as by the power of the facts that resisted introduction into this artificial system.

It is now nearly ten years since the theory of cyclones began to totter, and especially under the influence of a memoir published by Hann.† Following soon upon this the idea was developed by von Helmholtz‡ (*Berlin Sitzungsb.*, 1888), "that in the atmosphere, by continuously acting forces, the formation of surfaces of discontinuity is possible, and that the anticyclonic motion of the lower strata and the extensive, gradually increasing cyclone of the upper strata that are to be expected around the north and south poles, resolve themselves into a great number of *irregularly* progressing cyclones and anticyclones with an excess of the former." Von Helmholtz further says§ (*Berlin Sitzungsb.*, 1889): "if as in the case of the earth, the lower stratum is the denser, then it can be shown that the disturbances must at first proceed as do the waves of water raised by the wind." In this way the origin of cyclones and anticyclones is traced back to the general atmospheric circulation which is itself dependent on the difference in temperature between the equator and the poles; but the cyclones and anticyclones when once formed can still be considered as independent phenomena complete within themselves.

---

* Translated from *Verhandlungen der Gesellschaft Deutscher Naturforscher und Aerzte*, pp. 42–50 and 323–324, by Professor CLEVELAND ABBE.

† Comp. von Bezold, *Berliner Sitzungsberichte*, math.-nat. Kl., 1890, p. 831.

‡ Translated at pages 92, 93 of the *Mechanics of the Earth's Atmosphere.*

§ Translated at page 94 of the *Mechanics of the Earth's Atmosphere.*

The proof of this proposition was obtained by von Helmholtz by the application of the so-called theorem of the conservation of areas to a specific *portion* of the atmosphere, *i. e.*, a ring of air. But the application of this theorem in this manner is incorrect; it can only be applied to a free system of bodies (See *Met. Zeit.*, 1894, p. 114). Von Helmholtz applies this theorem to a ring of air cut out of the atmosphere; now such a ring of air does not form a free system but is subject to the conditions of its connection with the rest of the atmosphere, and the theorem of the conservation of angular momentum, or of areas, is not applicable to it. The application of this theorem to a ring-shaped portion of the atmosphere is only allowable in case the conditions within the atmosphere are of such a nature that they correspond entirely to the conditions at the boundary surface of a fluid and any other body. The conditions for the boundary surface of a fluid are: equal pressure on both sides of the boundary surface and equal values for the normal component of the velocities in the two masses that come in contact at the surface. (See Kirchhoff, *Mechanik*, 2d ed. 1877, p. 165; 3d ed. 1883, p. 164). These conditions are not fulfilled in the earth's atmosphere, which is in motion and not in static equilibrium. Therefore, that theory of atmospheric motions that depends upon their analysis into cyclones and anticyclones is deceptive and not so well established as it was believed to have been by the investigations of von Helmholtz.

Now, in order to find the point at which the prevailing views as to the motions of the atmosphere diverge from the reality it will be necessary to follow up a train of thought based upon indubitable well-established mechanical theorems. In doing this we will, as usual, begin with the consideration of an atmosphere that, without friction, surrounds the ideal figure of the earth, *i. e.*, a homogeneous ellipsoid of rotation, and whose particles are attracted by the mass of this ellipsoid, according to the general law of attraction of masses. It will be further assumed that, for a certain initial temperature, uniform throughout all its strata, the atmosphere has arrived at a uniform rate of revolution about the axis of rotation of the ellipsoid and with a velocity such that the ellipsoid represents the figure of equilibrium of an incompressible fluid.

Let this initial equilibrium of the atmosphere be now disturbed by introducing a different distribution of temperature, viz., one depending only on altitude above the ideal surface of the earth and on latitude, but uniform along any circle of latitude and every circle concentric therewith.

The *first* question now is whether a new state of equilibrium can develop, perhaps by means of a new rate of rotation that shall be different in the various strata of air vertically above each zone of latitude but be uniform in each circle concentric with and lying in the same plane as the circle of latitude.

The conditions for the equilibrium of a rotating fluid are:

$$\mu(R+r\omega^2)=\frac{\partial p}{\partial r}, \quad \mu Z=\frac{\partial p}{\partial z},$$

where the axis of z is the axis of rotation; μ is the density of the element of mass; r is the distance of the element of mass from the axis of z; ω is the angular velocity; R and Z are the components of the forces parallel to r and z; p is the elastic pressure. Let V be the potential of the attraction of the mass of the terrestrial ellipsoid; T the absolute temperature of the element of fluid mass. Let the characteristic equation of elasticity for gases be:

$$\frac{p}{T}=A\mu\,, \tag{1}$$

The equations of equilibrium can now be transformed into:

$$\frac{1}{T}\left(\frac{\partial V}{\partial r}+r\omega^2\right)=\frac{A}{p}\frac{\partial p}{\partial r}, \quad \frac{1}{T}\frac{\partial V}{\partial z}=\frac{A}{p}\frac{\partial p}{\partial z}\,.$$

By integrating the second equation there results:

$$A\log p=\int\frac{1}{T}\cdot\frac{\partial V}{\partial z}dz+F(r). \tag{2}$$

By differentiating this with respect to r, substituting the value of $\frac{A}{p}\cdot\frac{\partial p}{\partial r}$ thus obtained in the first equation and executing a partial integration we obtain:

$$r\omega^2=T\left\{\int\left(\frac{\partial V}{\partial z}\cdot\frac{\partial\frac{1}{T}}{\partial r}-\frac{\partial V}{\partial r}\cdot\frac{\partial\frac{1}{T}}{\partial z}\right)dz+F'(r)\right\}. \tag{3}$$

$F(r)$ and $F'(r)$ are determined by the surface conditions, in the present case by the circumstances existing in the highest layer of the atmosphere in which a uniform temperature and a velocity of rotation equal to that

of the earth will be assumed. Thus, p, μ and ω are determined by the preceding equations. Under the fundamental assumption that no other external forces than the attraction of the mass of the ellipsoidal earth are acting on the atmosphere, it follows that the changes in the distribution of temperature that produce changes in the motions of the atmosphere can produce no changes in the total moment (M) with reference to the axis of z. Therefore, a state of equilibrium can be brought about in the above-mentioned manner only for such distributions of temperature in the atmosphere as, by means of the above equations, give values for μ and ω that shall satisfy the equation

$$\int \mu . r^2 . \omega . dv = M, \tag{4}$$

the integration extending over the whole volume of the atmosphere. Therefore, a condition of equilibrium can only exist when the temperature T satisfies the equations (1), (2), (3) and (4). Such a condition of equilibrium cannot occur with every distribution of temperature, but, in general, there must exist movements in the direction from the equator to the pole, and inversely, besides a different rate of rotation for the different strata of air.

The *second* question now becomes this: Can a steady condition of motion occur in the atmosphere with an unvarying distribution of temperature? That is to say, under the assumed conditions, can the movement at each individual point in the atmosphere remain always the same?

Under the original assumption that the initial angular velocity is the same throughout the whole atmosphere, and that the temperature changes only with the distance from the pole, and with the altitude above the ideal surface of the earth, it must happen that whenever any steady condition does occur, the conditions must be the same at every point of any circle described about the axis of rotation. Therefore, throughout such a circle there can be no change of pressure with reference to either time or space [*i. e.* neither chronological nor geographical].

Such a distribution of pressure, however, cannot coexist with motions having a meridional component. This can be proved as follows:

Assuming that the temperature in the atmosphere depends only on the latitude and altitude, it has been shown that, in the steady condition, the motion and pressure must be the same for equal latitudes and altitudes. If, therefore, meridional components of motion exist, the individual

particles of air must move on closed surfaces of revolution about the earth's axis. Such a surface will contain, at any moment, all those particles that at any previous time crossed over any particular parallel circle of this surface; and the meridional component of motion must have the same direction along any meridional section of the surface. The space enclosed by one of these surfaces may enclose a part of the axis of rotation or not. The *first* case cannot occur, because on account of the general agreement in direction of the meridional component of motion, there would arise a steadily increasing accumulation of atmosphere at one pole of the surface of revolution, while from the opposite pole the adjacent air would steadily depart without any corresponding replacement by other air.

In the *second* case when the motion of the air occurs on a closed surface that does not intersect the earth's axis, we may refer back to the equation of continuity for fluid motion. This equation, referred to rectangular coördinates and for the case of steady motion, reads

$$\frac{\partial \mu \frac{dx}{dt}}{\partial x} + \frac{\partial \mu \frac{dy}{dt}}{\partial y} + \frac{\partial \mu \frac{dz}{dt}}{\partial z} = 0.$$

Transform this equation to another system of coördinates r, φ, z, connected with x, y, z by the equations

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z;$$

take the origin of coördinates at the centre of the earth, the axis of z as the earth's axis of rotation and recall that under the conditions here assumed, the derivatives with respect to φ are equal to zero, and we obtain:

$$\frac{\partial \mu \frac{dr}{dt}}{\partial r} + \frac{\partial \mu \frac{dz}{dt}}{\partial z} + \frac{\mu}{r} \cdot \frac{dr}{dt} = 0;$$

whence it follows that

$$\mu r \frac{dr}{dt} = -\frac{\partial F(r,z)}{\partial z}, \qquad \mu r \frac{dz}{dt} = +\frac{\partial F(r,z)}{\partial r}$$

and

$$\frac{\partial F(r,z)}{\partial z} \cdot \frac{dz}{dt} + \frac{\partial F(r,z)}{\partial r} \frac{dr}{dt} = 0.$$

But a function $F(r,z)$ varying continuously in such a way that the equation $F(r,z) = C$ shall represent closed sur-

faces lying only within or without a semi-ellipsoidal shell, does not exist. Therefore in no case, when the distribution of temperature causes meridional components of motion can a condition of steady motion and stationary pressure exist within the atmosphere.

Excepting the memoirs of von Helmholtz, already referred to, the assumption of a stationary condition has hitherto been adopted in all investigations into the motions of the atmosphere, *e. g.*, Ferrel, "Meteorological Researches," Part I., 1877, and Oberbeck (*Berlin. Sitzungb.* 1888).* Certain errors in the two memoirs by myself (*Met. Zeits.*, 1893, pp. 1, 131) also result from this erroneous assumption. Such an assumption has certainly led to impossible conclusions, but the fundamental nature of the error has, so far as I know, not been recognized hitherto.

Understanding, therefore, that in an atmosphere under these ideal assumptions, a steady condition cannot occur, it follows that the movements of the real atmosphere under conditions that imply continuous values and such as depend only on the latitude and the altitude above the earth's surface, must consist of oscillations and *regularly* progressing waves. These waves have no relation to the billows considered by von Helmholtz, for the existence of these latter depends on the formation of surfaces of discontinuity whereas the present discussion is based wholly upon continuous conditions of motion and pressure.

The waves originating in the difference of temperature between the equator and the poles may be classified into two groups. The first group includes waves that advance in the direction of the meridian, and that may partly resolve into standing oscillations with nodes at the poles; the second group consists of waves that advance in the direction of the circles of latitude. The atmospheric pressures and motions resulting from the combination of these two groups of intersecting waves, give rise to the phenomena hitherto called cyclonic and anticyclonic.

If the heights of the waves that advance in the direction of the circles of latitude vary with the latitude, then this would cause so-called cyclonic and anticyclonic phenomena in the atmosphere, even though no waves moving along the meridians were formed. In any case, however, the preceding considerations lead to the following conclusion: A large part of the atmospheric phenomena called cyclones and anticyclones, having a progressive motion along the earth's sur-

* Translated in the "Mechanics of the Atmosphere," pp. 176, 178.

face, belongs to the general atmospheric circulation that develops in consequence of the differences of temperature between the equator and the poles; that is to say such phenomena would be found in the terrestrial atmosphere even if there were no differences of temperature along the individual latitude circles and no friction or resistances of any kind. These phenomena are therefore not to be considered as local disturbances of an equilibrium of the atmosphere produced by the temporary distribution of temperature between the equator and the poles, but present themselves as the regular, normal and mechanically necessary phenomena of an atmosphere for which, in general, no such state of equilibrium can exist except in very definite and restricted cases.

In several important memoirs (*Vienna Sitzungsb., math. nat. Klasse*, vol. 99, Abt. IIa, p. 204; Vol. 101, Abt. 2a, p. 597; Vol. 102, Abt. 2a. p. 11 and p. 1369.) M. Margules has treated of atmospheric waves, by submitting to analytical investigation the *possible motions* of the air in a rotating spheroidal shell under certain assumptions made in order to simplify the calculations. But the results of Margules's investigations ought not to be applied without further consideration to an atmosphere of great height. According to Margules in the case of an advancing wave in a frictionless medium the strongest wind should occur simultaneously with the extremes of pressure.

Under the assumption, as before, of no friction and a distribution of temperature dependent only upon latitude and altitude there will be formed, besides the waves, a stationary zonal distribution of pressure corresponding to atmospheric currents parallel to the circles of latitude. This zone partially balances the disturbance of equilibrium due to a part of the temperature distribution, and must respond to the conditions under which equilibrium can occur in the atmosphere. If the meridional section shows that there are maxima and minima in this distribution of pressure, as appears to be actually the case, then these, in connection with the waves running along the zone of latitude, also lead to the formation of anticyclonic and cyclonic phenomena.

There have been many attempts to explain or compute this stationary distribution of pressure, as it is presented to us, mixed up with another pressure distribution due to thermal anomalies, in the mean values of atmospheric pressure, and that, too, in the belief that in these mean pressures, we have the simple result of the difference of temper-

ature between the equator and the poles. The attempts to deduce this evident zonal distribution of pressure, even in its general features, from general dynamic considerations must even *a priori* be considered as failures. It is as impossible for general considerations to lead to this result as for them to determine the form of an uniformly rotating fluid, whose particles are subject only to the general attraction of masses, as that of an ellipsoid.

The mathematical figure of the rotating fluid and of the surfaces of equal pressure in our atmosphere (therefore also the location and value of the zonal maxima and minima of pressure on the earth's surface, where it intersects the surfaces of equal pressure), can be found if not by observation, then only by expressing numerically the relations to each other of the quantities that come into consideration.

Such numerical computations of zonal distributions of pressure have been executed especially by Ferrel and Oberbeck. Both these authors, as already remarked, assume a stationary condition (as to temperature, motion and pressure). This would, perhaps, not prevent their computation of the stationary zonal part of the distribution of pressure from being, at least, approximately satisfactory. But Ferrel applies the theorem of the conservation of areas to an individual particle of air, which is not allowable; Oberbeck assumes the surface of the earth to be spherical, and neglects the differences of density in the atmosphere—which are precisely those that bring about the motions of the air.* Hence the problem to determine numerically the stationary zonal distribution of pressure is not yet resolved.

The wave motion, existing even in an ideal atmosphere, will be composed of groups of waves of different lengths and altitudes, as well as of different periods. In reference to the waves moving along the latitude circles, it is easy to see that the length of these must be a simple fraction of the whole circumference of the circle. In this case the pressure and motion correspond to the sum of a stationary zonal distribution, and of that due to the individual waves in their temporary positions. The progressive movement of the individual maxima and minima is therefore dependent upon the progress of the individual waves; the variations in the

*Oberbeck neglects the changes of temperature that depend upon the time in order to confine himself to steady motions and stationary conditions; he also considers the atmosphere as an incompressible liquid, thereby neglecting vertical differences of density due to pressure, but he does not neglect the horizontal differences of density due to temperature. C. A.

depths of the minima and heights of the maxima, as well as the distribution of pressure in their neighborhood, depend on the differences in phase that the waves have with respect to each other at the places where the minima or maxima occur. Therefore, for example, the minimum deepens when the deepest parts of the different waves approach each other. It is easy to see that the isobars that surround the minima and maxima do not in general have a circular form. This form can only occur when the differences of pressure in the direction from west to east are the same as those that simultaneously occur in the direction from north to south, a case that can only seldom occur, as is confirmed by experience, and which, in consequence of the further progress of the waves, is soon converted into other relations. Hence it is seen that when circular cyclonic and anticyclonic phenomena are made the foundation of our views as to atmospheric motions and other forms of isobars are considered as departures from these normal conditions—this is not accordant with the actual process of nature.

In the atmosphere surrounding the earth, on account of the irregular distribution of water and land, the zonal distribution of atmospheric temperature hitherto considered does not exist, and, consequently, the actual distribution of the pressure and the movement of the air will differ from that corresponding to such a zonal distribution of temperature. It will, however, be proper to analyze the relations actually existing into two components, one corresponding to the mean temporary zonal distribution of temperature and the other depending on any departure from this mean temperature over a greater or smaller region.

The phenomena resulting from this latter departure, even when they extend over large regions, still have a local character and are dependent on the nature of the respective terrestrial surfaces beneath them, and will, therefore, not take part in those motions that are caused by the average zonal temperature conditions but will be added to or superposed upon them. The analysis of the phenomena into these two subdivisions appears so much the more justifiable since the local temperature irregularities diminish with altitude, and the zonal temperature distribution becomes by so much the more important. Under circumstances that are easy to perceive, this summation or superposition of the progressive waves due to the general circulation, and of the pressure and motion due to local irregularities of temperature, can also produce progressive cyclonic and anticyclonic phenomena.

It must not be forgotten that the movements depending on the local irregularities of temperature cannot remain perfectly stationary but must also contain undulatory phenomena, for the pressure and motion cannot in general assume such a status that the motion of the particles of air follows the intersections of the surfaces of equal partial pressure with the level surfaces due to the force of terrestrial gravitation.

But this latter constitutes the general condition by reason of which the motion of the air in the rotating atmosphere may remain the same, that is to say steady, in any finite region and during any finite time. Now this condition is, in general, not fulfilled in the atmosphere (neither by a purely zonal distribution of temperature, nor by the partial motions that result from local departures of temperature from the zonal averages), and, hence winds that are steady as to direction and force cannot persist during any considerable interval of time, but atmospheric billows or even waves of small length must be formed. The preceding deductions lead us to the variations of pressure and the gusts within storms that have lately been demonstrated by observations, even for great altitudes, especially by Vallot on Mont Blanc and by Pernter on Sonnblick.

These latter and the atmospheric waves that become visible in the cloud formations known as "mackerel" sky, are probably the shortest waves that we have occasion to observe. The atmospheric waves of next greater extent are those that play an important role in the formation of thunderstorms, and a remark by Erk (*Met. Zeit.* 1894, p. 271) in reference to the thunderstorm of June 7, 1894, offers an excellent illustration of these. Of about the same order of dimensions are the tornadic storms whose great intensity depends upon the conjunction of several wave troughs. From the point of view here presented, we can also understand how it happens that such phenomena disappear and again reappear at other points in the prolongation of the direction of their earlier paths. This phenomenon can be regarded as a consequence of the interference of systems of waves progressing with different velocities.*

We must, therefore, think of the atmosphere as filled with numerous systems of waves of extremely different sizes and especially, first, the regular waves that correspond to a certain average zonal distribution of temperature and, second, the irregular waves that result from the local

*I must consider this explanation as wholly inapplicable to the tornadoes of the United States. C. A.

deviations from this distribution of temperature and which extend upward to only a slight altitude above the earth's surface.

As another class of progressive changes of atmospheric pressure, there must be mentioned certain sudden changes of pressure whose initial development must be ascribed to the formation of discontinuities in the atmosphere. Although, as before stated, the formation of surfaces of discontinuity in the atmosphere, as imagined by von Helmholtz as an application to a ring of air of the theorem of conservation of areas, does not correspond to the principles of mechanics, still other possibilities can be imagined that would lead to such discontinuities, as, for example, the sudden precipitation of the aqueous vapor contained in any portion of the atmosphere. But such discontinuities will always extend over small regions only. The sudden change of pressure due to these can, indeed, continue and advance like a wave produced by concussion, even when the original cause of the development of the discontinuity itself has disappeared. The phenomenon described in the *Annalen d. Hydrographie*, 1889, p. 242, and which was traced from Sylt to Pola may belong to this class.

Since we, at present, do not possess a reliable mathematical or analytical presentation of the atmospheric pressures and motions, even under the simplest assumptions, therefore, the problem that first occurs is to analyze the atmospheric relations as derived from observations into those more regular features that belong to the general atmospheric circulation and those due to local departures and to investigate each of these by itself.

The more accurate knowledge of the regular phenomena of the general atmospheric circulation will reveal to us the connection of phenomena that are, geographically speaking, widely separated from each other, and will lead us to certain periodicities in the weather. It will thus be seen whether, besides the regular seasonal variations in the influence of the solar heat on the earth, there be still other causes lying outside of our planet that affect the terrestrial atmosphere.

Thus the question will then be answered, to what extent the solar spot period, by a change in radiant heat or in electrical phenomena, affects the processes in our atmosphere. For it may here be stated that within any rotating fluid envelope, whose parts are attracted, according to the general law of the attraction of masses, by a solid of revolution within the fluid, there cannot exist any stationary con-

dition, or steady motion, when any influences, distributed zonally, prevent a rotation with uniform average, angular velocity of the whole envelope, or at least of those parts that are at equal distances from the axis of rotation. If, therefore, such influences exist on the sun, then wave formations must occur on it, and the sun spots, which have already been considered as vortex phenomena, as well as their variability, would thus find an explanation similar to that of the cyclones and anticyclones of the terrestrial atmosphere. It thus becomes quite possible that the regular phenomena of the general atmospheric circulation of the earth may, for purely mechanical reasons, have the same period as those on the sun. In this case there would not need to exist any direct connection between the sun spots and our weather, even though the same period should be established beyond doubt by observation. On the other hand, a further development of the theory may, perhaps, lead to conclusions as to certain mechanical relations, especially as to the velocity of rotation of the sun.

Even the discussion as to the influence of the moon on our weather, which is still by no means settled, will, in this way, be brought to an end, for it must then be possible to satisfactorily show whether the otherwise regular processes of atmospheric circulation are influenced by the changes in the moon's position, especially by its motion in declination.

It is to be hoped that the method here indicated may lead meteorology out of the region of vacillating ideas that now control it into a broader field, and place it among the exact sciences, where everything is reduced to numerical computation, and thus, to an important extent, further its application to daily practice.

DEUTSCHE SEEWARTE, HAMBURG.

SOME POINTS IN THE ELEMENTS OF THE THEORY OF FUNCTIONS.

BY PROFESSOR W. F. OSGOOD.

I. *A New Definition of an Analytic Function.* Cauchy defined $f(z)$ to be an analytic function of z when $f(z)$ is continuous and

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

converges toward one and the same limit when Δz con-

verges in any manner toward 0. From this definition, coupled with a requirement concerning the derivative $f'(z)$, the fundamental theorem of his whole theory is obtained, namely: If $f(z)$ is analytic and $f'(z)$ continuous throughout a region, then $\int f(z)dz$ taken along any closed path lying in the region vanishes.

We may, however, equally well turn things about and take as definition the following:

DEFINITION: *$f(z)$ is an analytic function of z if $f(z)$ is continuous throughout a region and $\int f(z)dz$ taken along any closed path lying wholly within the region vanishes.*

For then the function

$$F(z)=\int_a^z f(z)dz$$

will be an analytic function according to Cauchy's definition, and hence by Cauchy's theory its derivative $f(z)$ will also be analytic.

Thus while Cauchy's definition is based on the infinite process of differentiation, the present definition is based on the infinite process of integration, the integral being defined as the limit of a sum. Let it here be emphasized that these two processes, in the conceptions and definitions on which they rest, are wholly coördinate with each other; for neither can it be claimed in this respect that it is more elementary than the other.

There is one essential point of difference between these definitions. Cauchy's definition is based on the behavior of the function *in a point;* this definition prescribes the behaviour of the function *throughout a region.* Weierstrass's definition of an analytic function is of the latter kind. Furthermore, Cauchy is obliged in order to get any theorems worth mentioning, to demand* that the *derivative* $f'(z)$ be continuous throughout a region. Here, that requirement is brought nearer home by being transferred to the definition and there applies to the *function* itself.

It seems reasonable to expect that from this definition some elementary theorems will be more directly deducible than from Cauchy's definition. As an example take the theorem that a uniformly convergent series of analytic functions defines an analytic function.†

* One may, if one pleases, incorporate this demand into the definition; it seems to me undesirable to do so. But even then that which is essential in the present observation would not be changed.

† This application of the definition was called to my attention by Prof. Bôcher.

Let the series be

$$f_1(z)+f_2(z)+\cdots\cdots$$

Then its value $f(z)$ is a continuous function $f(z)$ and it remains to show that $\int f(z)dz$ taken along any closed path vanishes. This follows at once from the uniform convergence; the series can be integrated term by term and the integral of each term along the path in question vanishes.

It is to be noticed that this proof belongs to the most elementary class of proofs, in that it calls for no explicit representation of the functions entering (*e. g.*, by Cauchy's integral or by a power series).

II. A fundamental theorem in the elements of the theory of functions of a complex variable is the following:

(A) *If $f(z)$ is single-valued and analytic at all points* of a certain region except at the point $z=c$, and if $f(z)$ does not become infinite at c, then $f(z)$ is analytic in c also, or can be made so by the assignment to $f(z)$ in the point c of the value that $f(z)$ approaches when z approaches c.*

This theorem forms the basis for the theorems relating to the behavior of a single valued function $f(z)$ in the neighborhood of a singular point, P, $f(z)$ being analytic at all points in the neighborhood of P, namely, either (1) the function

$$\varphi(z)=(z-c)^m f(z), \quad z+c; \qquad \varphi(c)=0$$

where m denotes a positive integer, is analytic in P; or (2) $f(z)$ comes arbitrarily near to any assigned value in every neighborhood of the point P.

The proofs of this theorem ordinarily given are either wrong or far less elementary than one could wish, in view of the nature of the theorem. It is the purpose of this paper to consider some of these proofs and to give a new one which is at the same time simple and elementary.

Riemann's Proof.—Riemann† first proves the following lemma: Let u be a harmonic function‡ at all points of a

* With the new definition of an analytic function given in I, by the expression "$f(z)$ is analytic in the point a" is to be understood that a lies within a region throughout which $f(z)$ is analytic.

† *Inaugural Dissertation*, Göttingen, 1851; each theorem is here stated in somewhat less general form than in the original.

‡ By *harmonic* is meant a function which together with its first and second derivatives is continuous and satisfies Laplace's equation: $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

region except P; let

$$\lim_{\substack{x=0\\y=0}} \rho \frac{\partial u}{\partial x}=0, \qquad \lim_{\substack{x=0\\y=0}} \rho \frac{\partial u}{\partial y}=0,$$

where ρ denotes the distance of the point (x, y) from P; and finally let u remain finite in the neighborhood of P; then u is harmonic in P also, or can be made so by assigning to u in the point P the value that u approaches when (x, y) approaches P in any manner. He then states the theorem (A), replacing the words, "if $f(z)$ does not become infinite at c" by "if lim $(z-c)\, w=0$ when z approaches c", the function $f(z)$ being denoted by $w=u+vi$; and proves it as follows. He forms the integral

$$U=\int_{(x_0, y_0)}^{(x, y)} (u\, dx-v\, dy)$$

where v denotes the conjugate function of u, shows that its value is independent of its path and that the function U thus defined, which is continuous at P, is by the theorem just quoted harmonic at P also. And since

$$u=\frac{\partial U}{\partial x}, \qquad v=-\frac{\partial U}{\partial y}$$

the truth of the statement is evident.

The first of these theorems is interesting as showing how little stress Riemann laid on a systematic development of the theory of the logarithmic potential (*i. e.*, of harmonic functions) as a basis for the theory of functions of a complex variable. From this point of view the theorem (A) would be deduced from the theorem (B): If u is harmonic at all points of a certain region except at the point P and if u does not become infinite at P, then u is harmonic in P also, or can be made so by the assignment to u in the point P of its limiting value.

The Pseudo-Riemann Proof.—Riemann's rigorous proof of (A) has fared hardly with his expounders, whose version of it is as follows. Form the function

$$\varphi(z)=(z-c) f(z), \qquad z \neq c; \qquad \varphi(c)=0$$

Then $\varphi(z)$ is analytic in the neighborhood of c and continuous at c, *therefore* (sic) $\varphi(z)$ is analytic at c and can be developed by Taylor's theorem:

$$\varphi(z)=a_0+a_1(z-c)+a_2(z-c)^2+\cdots$$

But $\varphi(c) = 0$, hence $a_0 = 0$ and

$$\varphi(z) = (z-c)f(z) = a_1(z-c) + a_2(z-c)^2 + \cdots$$

or
$$f(z) = a_1 + a_2(z-c) + \cdots$$

This proof was given in the second edition of *Durège's Theorie der Functionen*, 1873, and has not been changed in the translation that has just appeared of the fourth edition. Curiously enough the proof given in the first edition, 1864, was rigorous, or at least could readily have been made so,—it is not complete. It was the proof given below as the first elementary proof and it seems likely that this proof was the one given by Riemann in his lectures. But in the improvements made in the second edition it was replaced by the one just cited. *Harnack*, whose desire frequently exceeded his ability to be rigorous, gave a similar proof,* likewise uncorrected in the English translation. *Forsyth* has also fallen into this error.† *Biermann*, in his attempt to expound Weierstrass's theory of functions, gives this same proof;‡ this passage was sharply criticised in the *Fortschritte d. Math.*, vol. 19, 1887, p. 363.

Perhaps the reason for Durège's changing to the above proof is to be found in a paper by Roch, one of the earliest expounders of Riemann, which formed the second of a set of three papers on Riemann's theory, appearing in Schlömilch.†† In § 8 (vol. 8, p. 187) Roch expounds § 13 of Riemann's inaugural dissertation and there does one of two things: either he makes precisely the error above pointed out or he tacitly assumes Theorem (A). On some accounts the latter seems more likely to be the case; but this passage would easily account for Durège's proof.

Proof by Laurent's Theorem. This theorem says that if $f(z)$ is analytic at all points of a circular ring, *i. e.*, of a region included between two concentric circles with centres at c, then $f(z)$ can be expressed within this region in the form:

$$f(z) = \sum_{-\infty}^{\infty} a_\mu (z-c)^\mu$$

From this theorem the theorem (A) can be deduced. But this proof of (A), if given by Cauchy's methods, involves in addition to Cauchy's integral theorems the representation

Diff.- u. Integralrechnung, § 189.

†*Theory of Functions*, p. 52.

‡*Analytische Functionen*, p. 174.

†† *Zeitschrift f. Math. u. Phys.* vol. 8, pp. 12-26 and pp. 183-203, 1863 and vol. 10, pp. 169-194, 1865.

of $f(z)$ by a power series and some properties of such series; if given by Weierstrass's methods, is still less elementary, the establishment of Laurent's theorem in that theory coming at a comparatively late stage. This subject is discussed at some length by Pringsheim in the last number of the *Math. Annalen*, who develops a number of elementary theorems, inclusive of the theorem (A), by the aid of power series plus the infinite process known as the *mean value*, a process which is essentially a restricted form of integration, and thus combining with the method of power series that which is essential in complex integration for the theorems in hand, he gives Cauchy's proofs of theorems in power series without the aid of Cauchy's integral.

Elementary Proofs of (A).—*First Proof.* Let C denote any curve enclosing the point c and lying within the region in question, and let t be any point but c within C. Surround c by a small circle γ not including t. Then by Cauchy's theorem

$$f(t)=\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-t}+\frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-t}$$

The second of these integrals vanishes, for $f(z)\,/\,z-t$ remains finite in the neighborhood of c, and when the radius of γ approaches 0, the length of the path of integration approaches 0. But the value of the integral is constant. hence it must be 0. Thus the formula

$$f(t)=\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-t}$$

represents the function $f(t)$ at all points within C except c. But the integral on the right hand side of this equation defines a function of t that is analytic in c too. Hence the theorem.

Second Proof. I now proceed to give a second elementary proof of (A), making use of the definition of I. Let

$$\varphi(z)=(z-c)f(z), \qquad z\neq c; \qquad \varphi(c)=0.$$

Then $\varphi(z)$ is analytic at c too; for $\varphi(z)$ is continuous throughout the region about c and $\int\varphi(z)\,dz$ taken along any closed path of this region vanishes, as will now be shown. Integration by parts gives

$$\int (z-c)f(z)\,dz=(z-c)\,F(z)-\int F(z)\,dz$$

where

$$F(z)=\int_a^z f(z)\,dz$$

is continuous and as is readily seen single-valued. Thus the first term on the right taken along any closed path of the region is 0. As to the second term, consider any closed path not including c. Then $F(z)$ and $F'(z)=f(z)$ are both continuous throughout the enclosed region and hence the integral along the boundary of the region is 0.* Finally let the path include c and surround c by a small circle. Then $\int F(z)\,dz$ extended in the usual way over the path and the circle will vanish. But the contribution that the circle yields is readily seen to be null; and thus $\int F(z)\,dz$ along any closed path whatever vanishes. Hence $\varphi(z)$ is analytic at c too. Its derivative at c is

$$\lim_{z=c} \frac{\varphi(z)}{z-c} = \lim_{z=c} f(z)$$

Hence $f(z)$ is continuous at c and thus fulfills both conditions of the definition of an analytic function given in I, and the proof is complete.†

HARVARD UNIVERSITY, *March* 1896.

ON THE MOTION OF A HOMOGENEOUS SPHERE OR SPHERICAL SHELL ON AN INCLINED PLANE, TAKING INTO ACCOUNT THE ROTATION OF THE EARTH.

BY PROFESSOR ALEXANDRE S. CHESSIN.

The influence of the rotation of the earth on the motion of bodies on its surface has been the subject of most interesting experiments, of which those with falling bodies and those of Foucault with the pendulum and the gyroscop may be mentioned. We propose to give here another interesting illustration of that influence, namely, on the motion of a homogeneous sphere or spherical shell on a plane inclined to the horizon. We will identify the system of axes (X, Y, Z) with the *absolute*, the system (Ξ, Y, Z) with the *relative*

* Cf. Goursat's proof of this theorem, *Acta Math.*, vol. IV.; Harkness and Morley, *Theory of Functions*, p. 164.

† Hölder has given in the *Math. Annalen*, vol. 20, 1882, a proof very similar to this. He integrates the given function $f(z)$ twice and shows that the second integral is analytic at c too. Hence he infers that $f(z)$ is analytic at c. I had obtained the proof here given, however, before Hölder's proof was known to me.

space, and first establish the differential equations of relative motion of a spherical shell in general.

Let ξ_i, η_i, ζ_i be the relative coördinates of a point (i) of a moving body; M its mass; V_ξ, V_η, V_ζ the components on the axes Ξ, Υ, Z, of the principal vector of the *real* forces, *i. e.*,

$$(1) \qquad V_\xi = \sum_i F_{i\xi}; \quad V_\eta = \sum_i F_{i\eta}; \quad V_\zeta = \sum_i F_{i\zeta};$$

where $F_{i\xi}, F_{i\eta}, F_{i\zeta}$ are the components on the same axes of the *real* forces applied at a point (i) of the body; in like manner let $V_\xi', V_\eta', V_\zeta'$ and $V_\xi'', V_\eta'', V_\zeta''$ be the components of the principal vectors of the *fictitious* forces of moving space and of compound accelerations, *i. e.*,

$$(2) \qquad V_\xi' = \sum_i m_i j_{e,\xi,i}; \quad V_\eta' = \sum_i m_i j_{e,\eta,i}; \quad V_\zeta' = \sum_i m_i j_{e,\zeta,i};$$

$$(3) \qquad V_\xi'' = \sum_i m_i j_{a,\xi,i}; \quad V_\eta'' = \sum_i m_i j_{a,\eta,i}; \quad V_\xi'' = \sum_i m_i j_{a,\zeta,i};$$

where m_i denotes the mass of a point (i); $j_{e,\xi,i}, j_{e,\eta,i}, j_{e,\zeta,i}$ denote the projections on the axes Ξ, Υ, Z, of the acceleration of the moving space at the point (i) and $j_{a,\xi,i}, j_{a,\eta,i}, j_{a,\zeta,i}$ the projections on the same axes of the compound acceleration of that point; finally let R_ξ, R_η, R_ζ be the components of the principal vector of the reactions due to the conditions of constraint; then we can write down the following three differential equations of the motion of the centre of inertia (c) of the body:

$$(4)_1 \qquad M\xi_c'' = V_\xi - V_\xi' + V_\xi'' + R_\xi$$

$$(4)_2 \qquad M\eta_c'' = V_\eta - V_\eta' + V_\eta'' + R_\eta$$

$$(4)_3 \qquad M\zeta_c'' = V_\zeta - V_\zeta' + V_\zeta'' + R_\zeta$$

The motion of the body about its centre of inertia will be defined by three differential equations, containing the projections of the moment of momentum of the *absolute* motion on a system of moving axes, having their origin at the centre of inertia of the body but otherwise independent of the same. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ denote the new axes; $m_1^{(a)}, m_2^{(a)}, m_3^{(a)}$ be the components on these axes of the moment of *absolute* momentum about the centre of inertia of the body; M_1, M_2, M_3 be the components on the same axes of the moment of the *real* forces; $\Lambda_1, \Lambda_2, \Lambda_3$ the components of the moment of the forces of reaction; finally let $\omega_1, \omega_2, \omega_3$ be the components of the angular velocity (ω) of the moving system

$(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z})$. Then the three differential equations which we were to establish are:

$$(5)_1 \qquad \frac{dm_1^{(a)}}{dt} = \omega_3 m_2^{(a)} - \omega_2 m_3^{(a)} + M_1 + \Lambda_1$$

$$(5)_2 \qquad \frac{dm_2^{(a)}}{dt} = \omega_1 m_3^{(a)} - \omega_3 m_1^{(a)} + M_2 + \Lambda_2$$

$$(5)_3 \qquad \frac{dm_3^{(a)}}{dt} = \omega_2 m_1^{(a)} - \omega_1 m_2^{(a)} + M_3 + \Lambda_3$$

Let now the body be a homogenous spherical shell; let H be its moment of inertia about a diameter; $p^{(a)}, q^{(a)}, r^{(a)}$ the components on the axes $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ of its *absolute* angular velocity; then, as we know,

$$(6) \qquad m_1 = Hp^{(a)}\ ;\ m_2 = Hq^{(a)}\ ;\ m_3 = Hr^{(a)}.$$

Further, let $p^{(r)}, q^{(r)}, r^{(r)}$ be the components on the same axes of the *relative* angular velocity of the spherical shell; $m_1^{(r)}, m_2^{(r)}, m_3^{(r)}$ be the components of the moment of *relative* momentum about the center of inertia of the body; then

$$(7)_1 \qquad m_1^{(a)} = Hp^{(a)} = H(p^{(r)} + \omega_1) = m_1^{(r)} + H\omega_1$$

$$(7)_2 \qquad m_2^{(a)} = Hq^{(a)} = H(q^{(r)} + \omega_2) = m_2^{(r)} + H\omega_2$$

$$(7)_3 \qquad m_3^{(a)} = Hr^{(a)} = H(r^{(r)} + \omega_3) = m_3^{(r)} + H\omega_3$$

$$(8)_1 \qquad \frac{dm_1^{(a)}}{dt} = \frac{dm_1^{(r)}}{dt} + H\frac{d\omega_1}{dt}$$

$$(8)_2 \qquad \frac{dm_2^{(a)}}{dt} = \frac{dm_2^{(r)}}{dt} + H\frac{d\omega_2}{dt}$$

$$(8)_3 \qquad \frac{dm_3^{(a)}}{dt} = \frac{dm_3^{(r)}}{dt} + H\frac{d\omega_3}{dt}$$

Turning now to our problem, let OZ be the direction of the force of gravity; XOY the plane of the horizon; $\Xi O Y$ the inclined plane forming an angle α with the horizon; $O\Xi$ be the line of slope; β the angle between the vertical plane ZO Ξ and the meridian of the point O; the positive axis OZ be so taken as to make an acute angle with the direction of the force of gravity (*i. e.*, with OZ); finally let the axes $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ be taken parallel to the axes Ξ, Y, Z. Then we have

$$(9) \qquad \frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = \frac{d\omega_3}{dt} = 0$$

and therefore

$$(10)\qquad \frac{dm_1^{(a)}}{dt}=\frac{dm_1^{(r)}}{dt}\ ;\ \frac{dm_2^{(a)}}{dt}=\frac{dm_2^{(r)}}{dt}\ ;\ \frac{dm_3^{(a)}}{dt}=\frac{dm_3^{(r)}}{dt}$$

so that the equations (5) reduce to the following:

$$(11)_1\qquad \frac{dm_1^{(r)}}{dt}=\omega_3 m_2^{(r)}-\omega_2 m_3^{(r)}+M_1+\Lambda_1$$

$$(11)_2\qquad \frac{dm_2^{(r)}}{dt}=\omega_1 m_3^{(r)}-\omega_3 m_1^{(r)}+M_2+\Lambda_2$$

$$(11)_3\qquad \frac{dm_3^{(r)}}{dt}=\omega_2 m_1^{(r)}-\omega_1 m_2^{(r)}+M_3+\Lambda_3$$

The three equations (4) together with the three equations (11) determine the motion of the spherical shell relatively to the earth. The expressions for $\omega_1, \omega_2, \omega_3$ are readily found to be:

$$(12)_1\qquad \omega_1=\omega\,(\sin\lambda\sin\alpha+\cos\lambda\cos\alpha\cos\beta)$$

$$(12)_2\qquad \omega_2=\omega\cos\lambda\sin\beta$$

$$(12)_3\qquad \omega_3=\omega(\sin\lambda\cos\alpha-\cos\lambda\sin\alpha\cos\beta)$$

ω being the angular velocity of the earth, and λ being the latitude of the point O.

In the following we will neglect terms of the order ω^2 of smallness and higher, because the assumption that the force of gravity is constant within the whole space in which the motion of the spherical shell takes place in our example, is justified only if the results be required with an approximation up to terms of the order of ω and not higher. We then find:

$$(13)\qquad V_\xi=Mg\sin\alpha\ ;\quad V_\eta=0\ ;\quad V_\zeta=Mg\cos\alpha$$

$$(14)\qquad V_\xi'=V_\eta'=V_\zeta'=0$$

$$(15)\qquad V_\xi''=2M\omega_3\eta_c'\ ;\ V_\eta''=-2M\omega_3\xi_c'\ ;\ V_\zeta=-2M(\omega_1\eta_c'-\omega_2\xi_c')$$

Let R be the reaction of the plane on which the sphere is moving; F_1 and F_2 the components of the force of friction between the sphere and the plane; then

$$(16)\qquad R_\xi=F_1\ ;\quad R_\eta=F_2\ ;\quad R_\zeta=\mathrm{R}$$

and equations (4) become

$$(17)_1\qquad M\xi_c''=Mg\sin\alpha+2M\omega_3\eta'+F_1$$

$$(17)_2 \qquad M\eta_c'' = -2M\omega_3\,\xi_c' + F_2$$

$$(17)_3 \qquad 0 = Mg\cos\alpha - 2M\,(\omega_1\eta_c' - \omega_2\,\xi_c') + R$$

If a denote the radius of the *exterior* surface of the spherical shell we shall have

$$(18) \qquad M_1 = M_2 = M_3 = 0$$

$$(19) \qquad A_1 = -a\,F_2\,; \quad A_2 = a\,F_1\,; \quad A_3 = 0$$

and equations (11) become

$$(20)_1 \qquad \frac{dp^{(r)}}{dt} - \omega_3\,q^{(r)} + \omega_2\,r^{(r)} = -\frac{a}{H}\,F_2$$

$$(20)_2 \qquad \frac{dq^{(r)}}{dt} - \omega_1\,r^{(r)} + \omega_3 p^{(r)} = \frac{a}{H}\,F_1$$

$$(20)_3 \qquad \frac{dr^{(r)}}{dt} - \omega_2 p^{(r)} + \omega_1\,q^{(r)} = 0$$

To make our problem a determinate one, it is necessary to indicate the way in which the spherical shell moves on the plane. We will assume that *it rolls on the plane without gliding*. Then the relative velocity of the point of contact of spherical shell and plane is zero, which gives the two equations of condition

$$(21) \qquad \xi_c' + a\,q^{(r)} = 0\,; \quad \eta_c' - a\,p^{(r)} = 0$$

If now we eliminate F_1 and F_2 from the seven equations $(17)_1$, $(17)_2$, $(20)_1$, $(20)_2$, $(20)_3$, (21), we obtain five equations between the five unknown quantities ξ_c, η_c, $p^{(r)}$, $q^{(r)}$, $r^{(r)}$, namely the three equations (20), and (21) and the following two:

$$(22)_1 \qquad (H + Ma^2)\frac{dq^{(r)}}{dt} = -Mag\sin\alpha - (H + 2Ma^2)\omega_3 p^{(r)} + H\omega_1 r^{(r)}$$

$$(22)_2 \qquad (H + Ma^2)\frac{dp^{(r)}}{dt} = (H + 2Ma^2)\,\omega_3 q^{(r)} - H\omega_2 r^{(r)}$$

A rigorous integration of these equations, although very simple, is not necessary, since we neglect terms of the order higher than ω. It is therefore preferable to integrate by successive differentiation, retaining every time only terms of the order not higher than ω, which gives

$$(23)_1 \qquad p^{(r)}=p_0^{(r)}+\frac{(H+2Ma^2)\,\omega_2 q_0^{(r)}-H\omega_2 r_0^{(r)}}{H+Ma^2}t$$
$$-\frac{Mag\sin a\,(H+2Ma^2)}{2\,(H+Ma^2)^2}\omega_2 t^2$$

$$(23)_2 \quad q^{(r)}=q_0^{(r)}-\frac{Mag\sin a}{H+Ma^2}t+\frac{H\omega_1 r_0^{(r)}-(H+2Ma^2)\,\omega_2 p_0^{(r)}}{H+Ma^2}t$$

$$(23)_3 \qquad r^{(r)}=r_0^{(r)}+(\omega_2 p_0^{(r)}-\omega_1 q_0^{(r)})\,t+\frac{Mag\sin a}{2\,(H+Ma^2)}\omega_1 t^2$$

where $p_0^{(r)}$, $q_0^{(r)}$, $r_0^{(r)}$ are the initial values of $p^{(r)}$, $q^{(r)}$, $r^{(r)}$. If we assume that the initial relative angular velocity of the spherical shell is zero, the above formulas reduce to the following:

$$(24)_1 \qquad p^{(r)}=-\frac{Mag\sin a\,(H+2Ma^2)}{2\,(H+Ma^2)^2}\omega_2 t^2$$

$$(24)_2 \qquad q^{(r)}=-\frac{Mag\sin a}{H+Ma^2}t$$

$$(24)_3 \qquad r^{(r)}=\frac{Mag\sin a}{2\,(H+Ma^2)}\omega_1 t^2$$

The values given by (23) or (24) being substituted in (21) a new integration gives immediately

$$(25)_1 \qquad \xi_c=\xi_{c0}-aq_0^{(r)}\,t+\frac{Ma^2 g\sin a}{2(H+Ma^2)}t^2$$
$$-\frac{a[H\omega_1 r_0^{(r)}-(H+2Ma^2)\omega_2 p_0^{(r)}]}{2(H+Ma^2)}t^2$$

$$(25)_2 \quad \eta_c=\eta_{c0}+ap_0^{(r)}\,t+\frac{a[(H+2Ma^2)\omega_2 q_0^{(r)}-H\omega_2 r_0^{(r)}]}{2\,(H+Ma^2)}t^2$$
$$-\frac{Ma^2 g\sin a\,(H+2Ma^2)}{6(H+Ma^2)^2}\omega_2 t^3$$

We will give special attention to the case when $p_0^{(r)}=q_0^{(r)}=r_0^{(r)}=0$; if, besides, $\xi_{c0}=\eta_{c0}=0$ which can always be assumed by properly choosing the origin of the axes of coördinates, we obtain the following equations:

$$(26)_1 \qquad \xi_c=\frac{Ma^2 g\sin a}{2(H+Ma^2)}t^2$$

$$(26)_2 \qquad \eta_c=-\frac{Ma^2 g\sin a(H+2Ma^2)}{6\,(H+Ma^2)^2}\omega_2 t^3$$

i. e., the centre of the spherical shell describes a semicubic parabola.

In the case of a solid sphere $H=\frac{2}{5}Ma^2$ and

$$(27)_1 \qquad \xi_c=\frac{5}{14}g\sin\alpha\, t^2$$

$$(27)_2 \qquad \eta_c=-\frac{10}{49}g\sin\alpha\,\omega_3\, t^3$$

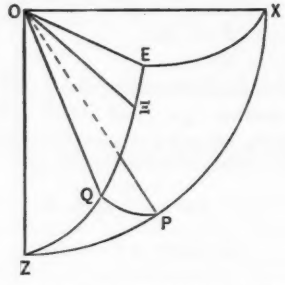
and therefore *the motion of the centre of a solid sphere does not depend on its mass or dimensions.*

We see that *the effect of the rotation of the earth on the motion of the centre of the sphere or spherical shell is to deviate it from the line of slope. This deviation may take place as well towards the east as towards the west of the meridian of the place* according as ω_3 is negative or positive, *i. e., according as* $\operatorname{tg}\lambda < \operatorname{tg}\alpha\cos\beta$ *or* $\operatorname{tg}\lambda > \operatorname{tg}\alpha\cos\beta$. *If* $\operatorname{tg}\lambda=\operatorname{tg}\alpha\cos\beta$ *the rotation of the earth has no influence on the motion of the centre of the sphere or spherical shell.*

These results may be interpreted geometrically as follows: Let OP be drawn parallel to the axis of the earth (toward the south) and let OQ be the projection of OP on the plane $\Xi\,0\,Z$. If then μ denote the angle of this projection with the horizon, we have

$$\operatorname{tg}\lambda=\operatorname{tg}\mu\cos\beta$$

Hence, if $\alpha<\mu$ *the deviation of the sphere or spherical shell from the line of slope takes place toward the west; if* $\alpha>\mu$ *this deviation takes place toward the east; and if* $\alpha=\mu$ *the sphere or spherical shell will move along the line of slope.*



$\sphericalangle XOE=\beta\ \sphericalangle EO\Xi=\alpha\ \sphericalangle EOQ=\mu\ \sphericalangle XOP=\lambda$

We have tacitly assumed that $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. However, the investigation of the different cases which may present themselves, although interesting, would carry us out of the limits of this paper. Let only one remark be made with respect to the variation of the angle β, namely, that *the motion of the sphere or spherical shell depends on the orientation of the inclined plane and the deviation from the line of slope is maximum when* $\beta=\pi$, *minimum when* $\beta=0$.

NOTES.

A REGULAR meeting of the American Mathematical Society was held in New York on Saturday afternoon, April 25, at three o'clock, Professor MANSFIELD MERRIMAN in the chair. There were twenty-one members present. On the recommendation of the Council, the following persons, previously nominated, were elected to membership: Mr. F. M. McGAW, Bordentown Military Institute, Bordentown, N. J.; Mr. A. B. CHACE, Valley Falls, R. I.; Professor E. D. ROE, Oberlin College, Oberlin, Ohio. Four nominations for membership were received. The following papers were read:

(1) Dr. EMORY McCLINTOCK: "On the most perfect forms of magic squares, with methods for their production."

(2) Professor CHARLOTTE ANGAS SCOTT: "On the theory of the birational transformation of a plane curve."

THE committee appointed by the Council to arrange for holding a colloquium in connection with the summer meeting at Buffalo, announces two courses of six lectures each, one by Professor Maxime Bôcher on the subject of Linear Differential Equations and their Applications, the other by Professor James Pierpont on the Galois Theory of Equations. The meetings of the Colloquium will begin immediately after the adjournment of the Society and will continue through the week. A special circular giving detailed information in regard to the arrangements for the Colloquium will be issued by the Committee early in July.

A NEW list of officers and members of the Society has been compiled by the Secretary and has been published by the Society. In the same pamphlet are included the Constitution and By-Laws of the Society, an account of the last Annual Meeting, and the reports of the Secretary and the Treasurer for 1895. Extra copies can be obtained from the Secretary.

PROFESSOR J. GRIESS, of Algiers, has translated into French Professor KLEIN's very interesting pamphlet, *Lectures on certain questions of elementary geometry*, reviewed by Miss SCOTT in the March number of the BULLETIN, pp. 157–164. The translator has expanded the text in a few places, and has substituted synthetic proofs for certain analytical proofs of the original. Nony & Company of Paris are the publishers of this translation.

COLUMBIA UNIVERSITY. The Department of Mathematics will give during the next academic year the following graduate courses, each course consisting of three lectures a week: by Professor T. S. Fiske, Special topics in the differential and integral calculus, first term; Differential equations and applications, second term; Theory of functions (second course), both terms. By Professor F. N. Cole, Theory of the complex variable, first term; Elliptic functions, second term. By Dr. J. B. Chittendon, Theory of invariants, second term. By Dr. J. E. Hill, Higher plane curves, first term; analytical theory of curves of double curvature and curved surfaces, second term. By Dr. G. H. Ling, Theory of numbers, both terms.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMERICAN MATHEMATICAL SOCIETY. See MATHEMATICAL PAPERS.

BOLZA (O.). See MATHEMATICAL PAPERS.

CATALOGO della insigne biblioteca appartenuta alla chiara memoria del principe Don Baldassare Boncompagni. 2a edizione. Parte I: Matematica, scienze naturali, ecc. Roma, 1895. 8vo. 511 pp.

CAUCHY (A.). Oeuvres complètes, publiées sous la direction scientifique de l'Académie des sciences et sous les auspices du Ministre de l'instruction publique, avec le concours de MM. Valson et Collet. 1re série, Vol. IX: Extraits des *Comptes rendus* de l'Académie des sciences. 1896. 4to. Fr. 25.00.

COLLET. See CAUCHY (A.).

DEL RE (A). Geometria projettiva ed analitica. Fascicoli 1 e 2. Modena, 1896. 8vo. pp. 1–256. Complete Fr. 8.50.

DUHAMEL (J. M. C.). Des méthodes dans les sciences de raisonnements. 3e édition. (En 5 parties). Partie II: Application des méthodes à la science des nombres et à la science de l'étendue. Paris, Gauthier-Villars, 1896. 8vo. Fr. 7.50.

EBNER (F.). Zur Theorie der Spiralflächen. [Diss.] Rostock, 1895. 8vo. 35 pp.

EULER (L.). Zwei Abhandlungen über sphärische Trigonometrie: Grundzüge der sphärischen Trigonometrie, und Allgemeine sphärische Trigonometrie, 1753–1779. Aus dem Französischen und Lateinischen übersetzt und herausgegeben von E. Hammer. Leipzig, 1896. 8vo. 65 pp. Cloth. Mk. 1.00

FORSYTH (A. R.). A treatise on the theory of functions of a complex variable. New York, Macmillan, 1896. 8vo. 22 and 682 pp. Cloth. $6.50

FREITAG (H.). Untersuchung: (I) der Kurve $(x/a)^{\frac{4}{3}}+(y/b)^{\frac{4}{3}}=1$, (II) der Fläche $(x/a)^{\frac{4}{3}}+(y/b)^{\frac{4}{3}}+(z/c)^{\frac{4}{3}}=1$. [Progr.] Schneeberg, 1896. 4to. 24pp.

GIUDICE (F.). See KLEIN (F.).

GRAHAM (J.). An elementary treatise on the calculus for engineering students, with numerous examples and problems worked out. London, Spon, 1896. 8vo. 266 pp.

HAMMER (E.). See EULER (L.).

HATHAWAY (A. S.). A primer of quaternions. New York, Macmillan, 1896. 12mo. 8 and 113 pp. Cloth. $0.90

HEIBERG (I. L.). See SERENUS ANTISSENSIS.

JORDAN (C.) Cours d' analyse de l'École polytechnique. 2e édition, entièrement refondue. Vol. III: Calcul intégral; équations différentielles. Paris, Gauthier-Villars, 1896. 8vo. 12 and 543 pp. Fr. 15.00

KLEIN (F.). L'œuvre géométrique de Sophus Lie. Traduit de l'anglais par L. Laugel. [*Nouvelles Annales de Mathématiques*, 3e série, Vol. 15, pp. 1–20].

———. Conferenze sopra alcune questioni di geometria elementare, redatte da F. Tägert, e tradotte dal tedesco da F. Giudice, con una prefazione di G. Loria. Torino, Rosenberg e Sellier, 1896. 8vo. 12 and 71 pp. Fr. 3.00

LATOON (F.). On common and perfect magic squares; with examples. London, 1896. 8vo. 140 pp. Cloth. 13s.

LAUGEL (L.). See KLEIN (F.).

LIE (S.) und SCHEFFERS (G.). Geometrie der Berührungstransformationen. Vol. I. Leipzig, Teubner, 1896. 8vo. 12 and 694 pp. Mk. 24.00

LORIA (G.). See KLEIN (F.).

MASCHKE (H.). See MATHEMATICAL PAPERS.

MATHEMATICAL PAPERS read at the International Mathematical Congress held in connection with the World's Columbian Exposition, Chicago, 1893. Edited by the Committee of the Congress: E. H. Moore, O. Bolza, H. Maschke, H. S. White. (*Papers published by the American Mathematical Society*, Vol. I.). New York, Macmillan, 1896. 8vo. 16 and 412 pp. Cloth. $4.00

MINKOWSKI (H.). Geometrie der Zahlen. (In 2 Lieferungen.) Lief. I. Leipzig, Teubner, 1896. 8vo. 240 pp. Mk. 8.00.

MOORE (E. H.). See MATHEMATICAL PAPERS.

NEWSON (H. B.). Continuous groups of projective transformations treated synthetically. 8vo. [*Kansas University Quarterly*, Vol. 4. 1895. pp. 71–92, 243–249.]

PAINLEVÉ (P.). See TISSERAND (F.).

PEANO (G.). Notions de logique mathématique. [*Association française pour l'avancement des sciences*, Congrès de Caen, 1894.] 8vo. 5 pp.

——— Sulla definizione di integrale. [*Annali di matematica pura ed applicata*, Vol. 23.] 4to. 4 pp.

POINCARÉ (H.). Remarques sur les fonctions abéliennes. Paris, Hermann, 1896. 4to. Fr. 4.00.

RICCI (F.). Le condizioni di divisibilità per alcuni numeri, dedotte da due teoremi generali dimostrati elementarmente. Firenze, Cirelli, 1896. 8vo. 7 pp.

ROMANO (S.). Sulla convergenza delle serie e metodi per determinare la somma. Noto, Zammit, 1896. 8vo. 38 pp.

SCHEELE (K.). Ueber die Auflösung algebraischer Gleichungen durch unendliche Reihen. [Progr.] Dresden, 1896. 4to. 29 pp.

SCHEFFERS (G.). See Lie (S.).

SERENUS ANTISSENSIS. Opuscula. Edidit et latine interpretatus est I. L. Heiberg. Leipzig, Teubner, 1896. 8vo. 20 and 303 pp. Mk. 5.00

STOLZ (O.). Grundzüge der Differential- und Integralrechnung. (In 2 Theilen.) Theil II: Complexe Veränderliche und Functionen. Leipzig, Teubner, 1896. 8vo. 10 and 338 pp. Mk. 8.00

TÄGERT (F.). See Klein (F.).

TISSERAND (F.). Recueil complémentaire d'exercices sur le calcul infinitésimal, à l'usage des candidats à la license et à l'agrégation des sciences mathématiques; avec de nouveaux exercices sur les variables imaginaires, par P. Painlevé. 2e édition. Paris, Gauthier-Villars, 1896. 8vo. 24 and 524 pp. Fr. 9.00

TRAUB (K.). Der verjüngte Magister matheseos. Ein Beitrag zur Sphärik und absoluten Geometrie. Lahr, Schauenburg, 1896. 8vo. 4 and 12 pp., 1 plate. Mk. 0.50

VALSON (C. A.). See CAUCHY (A.).

VERONESE (G.). Intorno ad alcune osservazioni sui segmenti infiniti e infinitesimi attuali. (*Mathematische Annalen*, Vol. 47, pp. 323–432.)

WHITE (H. S). See MATHEMATICAL PAPERS.

II. ELEMENTARY MATHEMATICS.

BIASI (G.). I poligoni equivalenti. Sassari, Dessi, 1896. 8vo. 4 pp.

BYERLY (W. E.). See CHAUVENET (W.).

CHAUVENET (W.). Plane Geometry, abridged and simplified by W. E. Byerly. New edition. Philadelphia, Lippincott, 1896. 8vo. Cloth. $0.40

EUCLID. See SACHAN (R.).

EXERCICES de géométrie en rapport avec les Éléments. Livre du maître, comprenant l'exposé des méthodes géométriques et 2000 questions résolues. 3e édition, revue et augmentée. Paris, 1896. 8vo. 20 and 1136 pp. Cloth. Fr. 16.00

FENKNER (H.). Arithmetische Aufgaben. Unter besonderer Berücksichtigung von Anwendungen aus dem Gebiete der Geometrie, Physik und Chemie. Für den mathematischen Unterricht an höheren Lehranstalten. Pensum der Unter-Tertia, Ober-Tertia und Untersekunda der neunstufigen, bezw. der Tertia, Sekunda und Prima der sechsstufigen Anstalten. 2te, auf Grund der Preussischen Lehrpläne vom Januar 1892 umgearbeitete Auflage. Braunschweig, Salle, 1894. 8vo. 250 pp. Mk. 2.20

FROLOV (M.). Démonstration de l'axiome XI d'Euclide. Paris, 1896. 8vo. 1 plate. Fr. 1.00

GANTZER (R.). Analogien aus der ebenen und körperlichen Trigonometrie. [Progr.] Magdeburg, 1896. 4to. 32 pp.

GÉLIN (E.). Recueil de problèmes d'arithmétiques. Huy, 1896. 8vo. 280 pp. Fr. 2.50

GRASS (J.). Die Veranschaulichung beim grundlegenden Rechnen. Erweiterte Ausgabe der Schrift über Gruppen-Zahlbilder. München, 1896. 8vo. 120 pp. Mk. 1.20

HERMES (O.). Verzeichnis der einfachsten Vielflache. [Progr.] Berlin, Gaertner, 1896. 4to. 24 pp., 1 plate.

HOLZMÜLLER (G.). Methodisches Lehrbuch der Elementar-Mathematik. Gymnasial-Ausgabe, in 2 Teilen. Im Anschluss an die Lehrpläne von 1892 nach Jahrgängen geordnet. Teil I: bis zur Abschlussprüfung der Untersekunda reichend. Leipzig, Teubner, 1896. 8vo. 8 and 228 pp. Mk. 2.40.

———. ——. Teil II: bis zur Entlassungsprüfung reichend. Leipzig, Teubner, 1896. 8vo. 8 and 280 pp. Mk. 3.00.

HOÜEL (J.). Tables de logarithmes à cinq décimales pour les nombres et les lignes trigonométriques suivies des logarithmes d'addition et de soustraction ou logarithmes de Gauss et de diverses tables usuelles. Nouvelle édition, revue et augmentée. Paris, Gauthier-Villars, 1896. 8vo. 48 and 118 pp. Fr. 2.00.

KOENIG (M.). Die geometrische Theilung des Winkels. Heft 2. Berlin, 1896. 8vo. 1 plate. Mk. 0.75

MATLEY (C. A.). Model solutions to certificate algebra and mensuration. London, Normal Press, 1896. 12mo. 56 pp. 1s. 6d.

MEDICI (C. DE). Medici's rational mathematics. (In 3 sections.) Section A, in 3 parts: Part I: The ABC of geometry, 20 pp.; Parts 2 and 3 (bound together): Object lessons in geometry, 56 pp. Section B, in 4 parts (bound together), 110 pp. New York, Lovell, 1895–96. 12mo. $1.50.

RANKIN (T. T.). Complete solutions to papers in mathematics (second stage) 1887–94, Science and art examinations. Second edition. London, 1896. 8vo. 104 pp. Cloth. 1s. 6d.

RECKNAGEL (G.). Ebene Geometrie. Lehrbuch mit systematisch geordneter Aufgabensammlung. 5te Auflage. München, 1896. 8vo. 8 and 222 pp. Mk. 2.00.

SACHAU (R.). The elements of Euclid, books 1 and 2. London, Arnold, 1806. 8vo. 184 pp. 1s. 3d.

SHUTTS (G. G.). See VAN VELZER (C. A.) and SHUTTS (G. G.).

SPIEKER (T.). Lehrbuch der Arithmetik und Algebra mit Uebungsaufgaben fün höhere Lehranstalten. 4te Auflage. Teil 2. Potsdam, Stein, 1896. 8vo. pp. 243–400. Mk. 1.40

STROMILLO (S.). Lezioni elementari di geometria per le scuole secondarie. Parte III. Napoli, Muca, 1896. 8vo. 165 pp. Fr. 2.00

TESTI (G. M.). Corso di matematiche. Vol. II. Geometria elementare. Livorno, Giusti, 1896. 8vo. 12 and 448 pp.

VAN VELZER (C. A.) and SCHUTTS (G. G.). Plane and solid geometry. Madison, Wis., Tracy, Gilbs & Co., 1896. 8vo. 8 and 395 pp.

III. APPLIED MATHEMATICS.

ATKINSON. See BOIS (H. DU).

BACH (C.). Die Maschinenelemente. Ihre Berechnung und Construction, mit Bücksicht auf die neueren Versuche. 5te, vermehrte Auflage. (In 2 Bänden.) Vol. I: Text; Vol. II: Tafeln und Tabellen. Stuttgart, 1896. 8vo. 18 and 662 and 29 pp., 53 plates. Mk. 28.00.

BOIS (H. DU). The magnetic circuit in theory and practice. Translated by Atkinson. London, Longmans, 1896. 8vo. 380 pp. 12s.

CARLI (A.). e FAVARO (A.). Bibliografia Galileiana 1568–1895 raccolta ed illustrata. (2108 opere disposte in ordine cronologico). Roma, 1895. 8vo. 512 pp.

CARRASSI DEL VILLAR (F.). La locomzione aerea basata sulle leggi della resistenza che un corpo in moto riceve dal mezzo nel quale questo corpo si muove; teoria analitica. Torino, Clausen, 1896. 4to. Fr. 5.00

CHAUDY (F.). Machines hydrauliques. Paris, Dunod, 1896. 16mo. 8 and 402 pp.

CUTTICA (R.). Manuale di celerimensura. Milano, 1896. 12mo. 8vo. 8 and 224 pp. Fr. 3.00

DARIÈS (G.). Cubature des terrasses et mouvement des terres, avec une préface de M. d'Ocagne. (Encyclopédie scientifique des aide-mémoire). Paris, Gauthier-Villars, 1896. Fr. 2.50

FAVARO (A.). See CARLI (A.) e FAVARO (A.).

FORBES (G.). Alternating and interrupted electric currents, being based upon three lectures delivered to the members of the Royal Institution. London, Biggs, 1896. 8vo. 102 pp. 2s. 6d.

GOUILLY (A.). Géométrie descriptive. Partie III: Changements de plans de projection, rotations, polyèdres. (Encyclopédie des aide-mémoire, section de l'ingénieur, No. 145 C.) Paris, Gauthier-Villars, 1896. 16mo. 157 pp. Fr. 2.50

HOLLENDER (H. J.). Ueber eine neue graphische Methode der Zusammensetzung von Kräften und ihre Anwendung zur graphischen Bestimmung von Inhalten, Schwerpunkten, statischen Momenten und Trägheitsmomenten ebener Gebilde. Leipzig, 1896. 8vo. 4 plates. Mk. 3.00.

HOUSTON (E. J.) and KENNELLY (A. E.). Electric Heating. New York, Johnston, 1896. 16mo. Cloth. $1.00

———. Electro-dynamic machinery. New York, Johnston, 1896. 16mo. 322 pp. Cloth. $2.50

———. Electro-magnetism. New York, Johnston, 1896. 16mo. 286 pp. Cloth. $1.00

JOHANNESSON (P.). Das Beharrungsgesetz. [Progr.] Berlin, Gaertner, 1896. 4to. 26 pp. Mk. 1.00

KÄUFFER (P.). Energie-Arbeit; die Kräftediagramme, die specifische Wärme der Luft (Gase); "Energie" im Allgemeinen, etc. Mainz, 1896. 8vo. 50 pp. Mk. 1.00

KENNELLY (A. E.). See HOUSTON (E. J.) and KENNELLY (A. E.).

KOCH (G.). Die Lösung des Flugproblems in physikalischer und maschinentechnischer Hinsicht. Mit einen Anhang: Nutzen und Folgen praktischer Luftschiffahrt. München, Lukaschik, 1896. 8vo. 72 pp. 8 plates. Mk. 3.00

LANNOY (G. DE). Précis de cosmographie et de navigation et notions de trigonométrie sphérique. Paris, 1896. 8vo. Fr. 6.00

LAUENSTEIN (R.). Die graphische Statik. Elementares Lehrbuch für technische Lehranstalten und zum Gebrauch in der Praxis. 3te Auflage, Stuttgart, 1896. 8vo. Mk. 4.00.

MACLEAN (M.). Physical units. London, Biggs, 1896. 8vo. 152 pp. 2s 6d.

MESSINA (A.). I misuratori di energia elettrica. Palermo, Reber, 1896. 8vo. 20 and 130 pp. Fr. 5.00.

MORGAN (S.). See RADFORD (C. S.).

MORTARA (E.). Osservazione sul principio delle aree. Parma, Battei, 1896. 16mo. 7 pp.

OCAGNE (M. D'). See DARIÈS (G.).

PEANO (G.). Sul moto del polo terrestre. (*Accademia reale delle scienze di Torino*, anno 1894–95.) 8vo. 11 pp.

PRESTON (E. D.). Telegraphic determination of the force of gravity at Baltimore, Md., from simultaneous pendulum observations at Washington and Baltimore. (*U. S. Coast and Geodetic Survey*, Report for 1894, Appendix 2.) Washington, Government Printing Office, 1895. 4to. pp. 58–70.

RADFORD (C. S.) Handbook of naval gunnery; prepared by the authority of the Navy Department for use in the U. S. Navy, U. S. Marine Corps, and States Naval Reserves. 2nd edition, revised and enlarged, with the assistance of S. Morgan. New York, Van Nostrand, 1896. 16mo. Leather. $1.50

RAYLEIGH (BARON). See STRUTT (J. N.).

SCHEURER-KESTNER (A.). Pouvoir calorique des combustibles solides, liquides et gazeux. Paris, 1896. 12mo. 14 and 288 pp. Fr. 5.00.

SPIES (P.). Wellen und Strahlen in ihrer Bedentung für die neuere Naturforschung mit besonderer Berücksichtigung der Versuche über elektrische Strahlungen. 2te, durch Zusätze über neuere Experimente ergänzte Auflage. Berlin, 1896. 8vo. 32 pp. Mk. 0.60

STRUTT (J. W.). The theory of sound. (In 2 volumes.) Vol. II. 2nd edition, revised and enlarged. London, Macmillan, 1896. 8vo. 520 pp. Cloth. 12s.

VOIGT (W.). Kompendium der theoretischen Physik. (In 2 Bändem.) Vol. II: Elektricität und Magnetismus; Optik. Leipzig, Veit, 1896. 8vo. 14 and 810 pp. Mk. 18.00

ZEPF (K.). Einführung in die Grundlehren vom elektrischen Strom Freiburg, Ragoczy, 1896. 8vo. 16 and 118 pp. 20 plates. Mk. 3.00